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# Unsteady, three-dimensional, boundary-layer flow due to a stretching surface

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## 1. INTRODUCTION

THE FLOW and heat transfer problem due to a stretching boundary is important in extrusion processes. Tsou *et al.* [1] and Crane [2] among others have studied the steady flow problem caused by the two-dimensional stretching of a flat surface. Recently, a number of authors [3–7] have studied various aspects of this problem. More recently, Wang [8] considered the steady three-dimensional flow due to a stretching flat plate, where only the velocity field was studied.

The aim of the present analysis (which is an extension of Wang [8]) is to study the flow, heat and species transport problem due to the unsteady, three-dimensional flow caused by the stretching of a flat surface in two lateral directions. A self-similar solution has been obtained when the flat surface is stretched in a particular manner. The resulting nonlinear ordinary differential equations have been solved numerically [9].

## 2. GOVERNING EQUATIONS

We consider a highly elastic membrane immersed in a viscous fluid which is continuously stretched in the $x$ and $y$ directions and which also varies with time (see Fig. 1). The fluid velocities on the surface ($z = 0$) are given by:

$$u_w = ax(1-\lambda t^*)^{-1}, \quad v_w = by(1-\lambda t^*)^{-1}, \quad t^* = at. \tag{1}$$

The fluid has no lateral motions at $z \to \infty$. Also, it is assumed to have constant properties, and both wall and free stream are maintained at uniform temperature and concentration. The viscous dissipation term has been neglected. Here, we can confine our analysis to species diffusion processes in which the diffusion-thermal and thermo-diffusion effects can be neglected. The interfacial velocity at the wall $w_w$ due to mass diffusion process has also been neglected in the analysis. Under the foregoing assumptions, the unsteady boundary-layer equations governing the flow, and heat and diffusion transport can be expressed as:

$$u_x + v_y + w_z = 0 \tag{2}$$

$$u_t + uu_x + vu_y + wu_z = \nu u_{zz} \tag{3}$$

$$v_t + uv_x + vv_y + wv_z = \nu v_{zz} \tag{4}$$

$$T_t + uT_x + vT_y + wT_z = \alpha T_{zz} \tag{5}$$

$$C_t + uC_x + vC_y + wC_z = DC_{zz}. \tag{6}$$

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The initial and boundary conditions are given by

$$\left.\begin{array}{lll} u(x,y,z,0) = u_i, & v(x,y,z,0) = v_i, & w(x,y,z,0) = w_i \\ T(x,y,z,0) = T_i, & C(x,y,z,0) = C_i & \end{array}\right\} \tag{7a}$$

$$\left.\begin{array}{lll} u(x,y,0,t) = u_w, & v(x,y,0,t) = v_w, & w(x,y,0,t) = 0 \\ T(x,y,0,t) = T_w, & C(x,y,0,t) = C_w & \end{array}\right\} \tag{7b}$$

$$\left.\begin{array}{ll} u(x,y,\infty,t) = v(x,y,\infty,t) = 0, & T(x,y,\infty,t) = T_\infty \\ C(x,y,\infty,t) = C_\infty. & \end{array}\right\} \tag{7c}$$

We apply the following transformations

$$\left.\begin{array}{lll} \eta = (a/\nu)^{1/2}(1-\lambda t^*)^{-1/2}z, & \lambda t^* < 1, & c = b/a \\ u = ax(1-\lambda t^*)^{-1}f'(\eta), & v = ay(1-\lambda t^*)^{-1}s'(\eta) & \end{array}\right\} \tag{8a}$$

$$\left.\begin{array}{lll} w = -(a\nu)^{1/2}(1-\lambda t^*)^{-1/2}(f+s), & Pr = \nu/\alpha, & Sc = \nu/D \\ (T-T_\infty)/(T_w-T_\infty) = g(\eta), & (C-C_\infty)/(C_w-C_\infty) = G(\eta) & \end{array}\right\} \tag{8b}$$

to equations (2)–(6) and we find that (2) is satisfied identically and equations (3)–(6) reduce to

$$f''' + (f+s)f'' - f'^2 - \lambda(f' + \eta f''/2) = 0 \tag{9}$$

$$s''' + (f+s)s'' - s'^2 - \lambda(s' + \eta s''/2) = 0 \tag{10}$$

$$Pr^{-1}g'' + (f+s)g' - \lambda\eta g'/2 = 0 \tag{11}$$

$$Sc^{-1}G'' + (f+s)G' - \lambda\eta G'/2 = 0. \tag{12}$$

The boundary conditions reduce to

$$\left.\begin{array}{ll} f = s = 0, \quad f' = g = G = 1, \quad s' = c & \text{at } \eta = 0 \\ f' = s' = g = G = 0 \quad \text{as } \eta \to \infty. & \end{array}\right\} \tag{13}$$

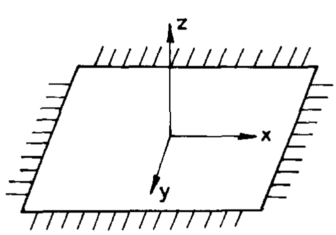


FIG. 1. Coordinate system.

## NOMENCLATURE

| | |
|---|---|
| $a, b$ | velocity gradients in the $x$ and $y$ directions, respectively |
| $c$ | ratio of the velocity gradients |
| $C, D$ | species concentration and binary diffusion coefficients, respectively |
| $C_{fx}, C_{fy}$ | skin friction coefficients in the $x$ and $y$ directions, respectively |
| $f, s$ | dimensionless streamfunctions |
| $f', s'$ | dimensionless velocity components in the $x$ and $y$ directions, respectively |
| $f''_w, s''_w$ | skin-friction parameters in the $x$ and $y$ directions, respectively |
| $g, G$ | dimensionless temperature and concentration, respectively |
| $g'_w, G'_w$ | heat transfer and mass flux parameters, respectively |
| $k, m_w$ | thermal conductivity and mass flux of the diffusing species |
| $Nu, Sh$ | local Nusselt and Sherwood numbers, respectively |
| $Pr, Sc$ | Prandtl and Schmidt numbers, respectively |
| $q_w$ | local heat transfer rate per unit area |
| $Re_x, Re_y$ | Reynolds numbers in the $x$ and $y$ directions, respectively |
| $t, t^*$ | dimensional and dimensionless times, respectively |
| $T$ | dimensional temperature |
| $u, v, w$ | velocity components in the $x, y, z$ directions, respectively |
| $x, y, z$ | principal, transverse and normal directions, respectively. |

Greek symbols

| | |
|---|---|
| $\alpha, \eta$ | thermal diffusivity and similarity variable, respectively |
| $\lambda$ | parameter characterizing the unsteadiness in the wall velocity |
| $\mu, \nu, \rho$ | coefficient of viscosity, kinematic viscosity and density, respectively |
| $\tau_x, \tau_y$ | shear stresses in the $x$ and $y$ directions, respectively. |

Superscript

| | |
|---|---|
| $'$ | differentiation with respect to $\eta$. |

Subscripts

| | |
|---|---|
| i | initial conditions |
| $t, x, y, z$ | derivatives with respect to $t, x, y, z$, respectively |
| w, $\infty$ | wall and free-stream conditions, respectively. |

It may be noted that many industrial processes involve the cooling of continuous strips by drawing through a quiescent fluid: the drawing of a sheet glass is an example of such a process. The properties of the final product depend to a large extent on the rate at which the material is cooled. The velocity components of the sheet may be taken as proportional to the distance and time as in equation (1).

It may be remarked that for $\lambda = 0$, equations (9)–(12) reduce to those of the steady-state case. However, only the velocity fields, i.e. only equations (9) and (10) with $\lambda = 0$ have been studied by Wang [8]. It is to be noted that equation (12) is the same as (11) if we replace $Sc$ by $Pr$ and $G$ by $g$. The boundary conditions are also identical. Here the parameter $c$ denotes the nature of the stagnation point and for nodal point flows $c \geq 0$ $(0 \leq c \leq 1)$. Also $c = 0$ for a two-dimensional stagnation point and $c = 1$ for an axisymmetric stagnation point. As most three-dimensional bodies of practical interest lie between a cylinder ($c = 0$) and a sphere ($c = 1$), the computations have been carried out for $0 \leq c \leq 1$.

The surface skin friction coefficients in the $x$ and $y$ directions can be expressed as

$$\left.\begin{aligned} C_{fx} &= 2\tau_{xw}/\rho_w^2 = -2(Re_x)^{-1/2} f''_w \\ C_{fy} &= 2c^2\tau_{yw}/\rho v_w^2 = -2c^{1/2}(Re_y)^{-1/2} s''_w \end{aligned}\right\} \quad (14a)$$

where

$$\left.\begin{aligned} \tau_{xw} &= -\mu(u_z)_w, & \tau_{yw} &= -\mu(v_z)_w \\ Re_x &= u_w x/\nu, & Re_y &= v_w y/\nu. \end{aligned}\right\} \quad (14b)$$

The local heat transfer coefficient in terms of Nusselt number is given by

$$Nu = xq_w/[k(T_w - T_\infty)] = -(Re_x)^{1/2} g'_w \quad (15a)$$

where

$$q_w = -kT_z. \quad (15b)$$

Similarly, the Sherwood number characterizing the mass flux of the diffusing species can be written as

$$Sh = (x/\rho D)[m_w/(C_w - C_\infty)] = -(Re_x)^{1/2} g'_w \quad (16a)$$

where

$$m_w = -\rho D(C_z)_w. \quad (16b)$$

## 3. RESULTS AND DISCUSSION

As mentioned earlier, equation (12) is identical to (11). Hence the two-point boundary-value problem represented by equations (9)–(11) under the relevant conditions given in (13) have been solved numerically [9] on a high speed computer (CDC CYBER 205). An outline of the method is presented in ref. [9] and hence for the sake of brevity it is not presented here. The effect of step size $\Delta\eta$ and the edge of the boundary layer $\eta_\infty$ on the solution has been studied to optimize them. The results presented here are independent of $\Delta\eta$ and $\eta_\infty$ at least up to fifth decimal place. The computations have been carried out for various values of the parameters. However, the results have been presented only for some representative values of the parameters.

In order to assess the accuracy of our method, we have compared our results ($f''_w, s''_w$) for $\lambda = 0$ with those tabulated by Wang [8] and found them in excellent agreement. They agree up to fourth decimal place. Hence the comparison is not shown here.

The effect of the parameter $\lambda$ characterizing the unsteadiness on the skin friction and heat transfer ($-f''_w, -s''_w, -g'_w$) is shown in Table 1 and Fig. 2. It is observed that the skin friction parameters in the $x$ and $y$ directions ($-f''_w, -s''_w$)

Table 1. Skin friction, heat transfer and mass diffusion parameters for $c = 0.5$, $Pr = 0.7$

| $\lambda$ | $-f''_w$ | $-s''_w$ | $-g'_w$ |
|---|---|---|---|
| −1.00 | 0.7912 | 0.2956 | 0.8053 |
| −0.75 | 0.8673 | 0.3384 | 0.7585 |
| −0.50 | 0.9430 | 0.3809 | 0.7068 |
| −0.25 | 1.0183 | 0.4232 | 0.6477 |
| 0 | 1.0931 | 0.4652 | 0.5758 |
| 0.25 | 1.1674 | 0.5059 | 0.4713 |
| 0.50 | 1.2407 | 0.5480 | 0.3401 |
| 0.75 | 1.3122 | 0.5878 | 0.1849 |
| 1.00 | 1.3814 | 0.6261 | 0.0109 |

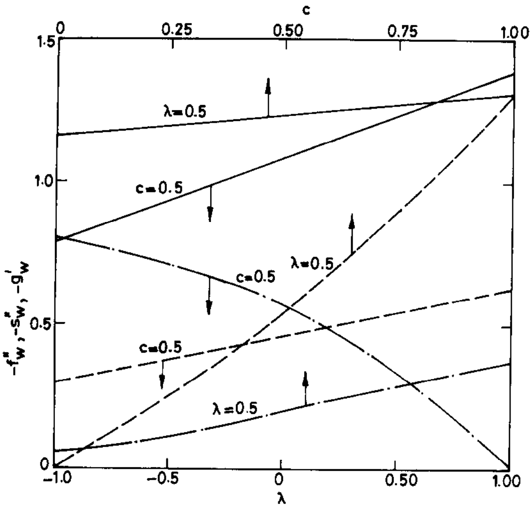


FIG. 2. Skin friction and heat transfer parameters $(-f''_w, -s''_w, -g'_w)$ for $Pr = 0.7$. —— $-f''_w$; --- $-s''_w$; —·— $-g'_w$.

Table 2. Heat transfer and mass diffusion parameters for $\lambda = c = 0.5$

| $Pr$ | $-g'_w$ | $Sc$ | $-G'_w$ |
|---|---|---|---|
| 0.2 | 0.1254 | 1.0 | 0.2446 |
| 0.7 | 0.2011 | 7.0 | 1.5641 |
| 1.0 | 0.2446 | 10.0 | 2.4769 |
| 7.0 | 1.5641 | 50.0 | 5.9954 |

increase as $\lambda$ increases, but the heat transfer $-g'_w$ decreases. This is due to the reduction in the momentum boundary-layer thicknesses and increase in the thermal boundary-layer thickness. It may be noted that a similar effect has been observed by Teipel [10] for unsteady forced convection flow at a three-dimensional stagnation point for a stationary wall. The effect of $\lambda$ on the mass flux of diffusing species $(-G'_w)$ is similar to that on the heat transfer $(-g'_w)$, hence not shown here. In fact for $Sc = Pr$, $g'_w = G'_w$.

The effect of the nature of the stagnation point characterized by the parameter $c$ on the skin friction and heat transfer has also been presented in Fig. 2. It is found that the skin friction and heat transfer parameters $(-f''_w, -s''_w, -g'_w)$ decrease as $c$ decreases. The effect of $c$ is more pronounced on $-s''_w$ and $-g'_w$ but its effect on $-f''_w$ is comparatively small. The effect of $c$ on $-G'_w$ is similar to that on $-g'_w$.

Table 2 shows the effect of the Prandtl number ($Pr$) on the heat transfer $(-g'_w)$ and the effect of the Schmidt number ($Sc$) on the mass flux of diffusing gases. $Pr$ or $Sc$ does not affect the skin-friction parameters $(-f''_w, -s''_w)$. It is seen that the heat transfer $(-g'_w)$ increases with $Pr$ because a higher Prandtl number fluid has a relatively lower thermal conductivity which reduces conduction and thereby increases the variations. This results in the decrease in the thermal boundary-layer thickness and increase in the convective heat transfer at the wall. The effect of the Schmidt number ($Sc$) on the mass flux of diffusing species $(-G'_w)$ is similar to that of the Prandtl number on the heat transfer $(-g'_w)$.

The velocity and temperature profiles $(f', s', g)$ for different values of $\lambda$ are shown in Figs. 3–5. These profiles decay exponentially as $\eta$ increases for all values of $\lambda$. The velocity profiles $(f', s')$ become steeper as $\lambda$ increases, but its effect on the temperature profile ($g$) is just opposite. The reason for such a behaviour has been explained earlier while discussing the effect of $\lambda$ on $f''_w, s''_w, g'_w$. The effect of $\lambda$ on the concentration profile ($G$) is similar to that on the temperature

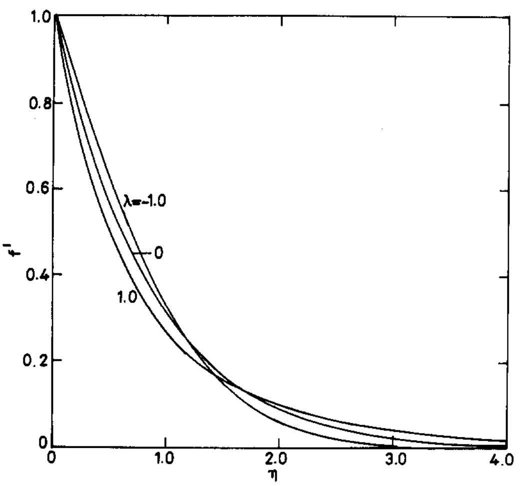


FIG. 3. Velocity profile in the $x$ direction $(f')$ for $c = 0.5$.

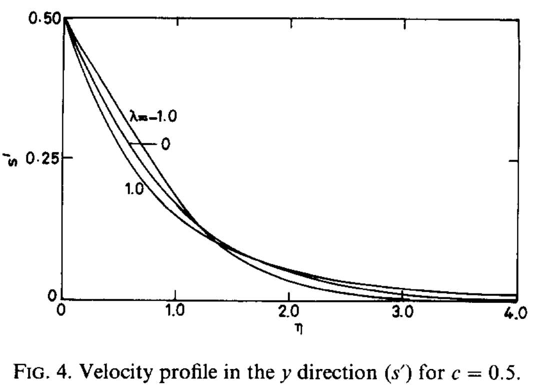


FIG. 4. Velocity profile in the $y$ direction $(s')$ for $c = 0.5$.

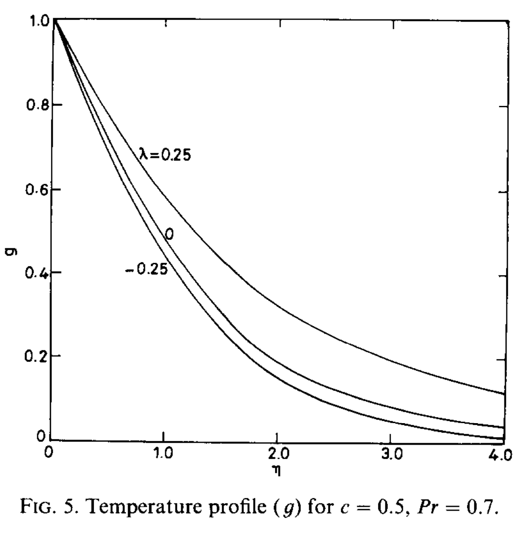


FIG. 5. Temperature profile ($g$) for $c = 0.5$, $Pr = 0.7$.

profile ($g$). It is also observed that the velocity profiles ($f'$, $s'$) at any two values of $\lambda$ cross each other towards the edge of the boundary layer. A similar trend has been observed by Yang [11] for the unsteady, two-dimensional, stagnation-point flow over a stationary wall.

## 4. CONCLUSIONS

The effects of the unsteadiness in the wall velocities and the nature of the stagnation point on the skin friction, heat transfer and mass flux of diffusing species are found to be appreciable. The Prandtl number and the Schmidt number strongly affect the heat transfer and mass flux of diffusing species, respectively. The velocity temperature and concentration profiles are observed to decay exponentially.

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# Heating or evaporation in the thermal entrance region of a non-Newtonian, laminar, falling liquid film

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## 1. INTRODUCTION

HEATING or evaporation of non-Newtonian solutions by means of falling film shell-and-tube heat exchangers is sometimes practised in the food and polymer processing industries. The application of the falling film principle has the advantage of short residence time which is most desirable for heat-sensitive materials. In short columns and when the viscosity of the solution is high, the film flow may be laminar in nature. Little information, however, is available on the heat transfer rate in these liquid films. Murthy and Sarma [1] investigated analytically heating in the entrance region of an accelerating, non-Newtonian, power-law-model, laminar falling film flowing down an inclined plane with constant wall temperature. Integral solutions for the boundary-layer equations of momentum and energy were obtained in which the Nusselt number for the thermally developing and fully developed regions can be calculated. Heating with constant wall temperature and a fully developed velocity profile was also analyzed both theoretically and experimentally by Stucheli and Widmer [2] for Newtonian and non-Newtonian power-law model falling film on an inclined plane. The viscosity was assumed to be temperature dependent. The objectives of the present research are to show that a simple analytical solution can be easily obtained for heating or evaporation in the thermal entrance and fully developed regions of a non-Newtonian, power-law model falling film with the boundary condition of constant wall heat flux or constant wall temperature.

## 2. THEORY

A non-Newtonian liquid film of average film thickness, $\delta$, is in steady laminar flow down a vertical plane under the action of gravity. The liquid flow is characterized by a power-law rheological model. The velocity profile of the falling film is assumed to be fully developed at the start of the heat transfer section. By a balance of shear and gravity forces, the dimensionless velocity profile can be derived, with the boundary condition of no slip at the wall ($y = 0$) and zero interfacial shear at the gas–liquid interface ($y = \delta$), as

$$U^*(\eta) = U(\eta)/U_1 = 1-(1-\eta)^{(n+1)/n} \tag{1}$$

where $\eta = y/\delta$ and $y$ is the distance measured from the wall into the liquid film with $x$ being the coordinate in the flow direction. The average velocity and surface velocity can be derived as

$$U_{av}/U_1 = (n+1)/(2n+1) \tag{2}$$

$$U_1 = \frac{n}{n+1}\left(\frac{\rho g}{K}\right)^{1/n} \delta^{(n+1)/n}. \tag{3}$$